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An Example of Nonuniqueness for Solutions to the Homogeneous Boltzmann Equation

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The paper deals with the spatially homogeneous Boltzmann equation for hard potentials. An example is given which shows that, even though it is known that there is only one solution that conserves energy, there may be other solutions for which the energy is increasing; uniqueness holds if and only if energy is assumed to be conserved.

KEY WORDS: Boltzmann equation; hard-potential interaction; nonuniqueness; nonconservation of energy.

1. INTRODUCTION

This paper deals with the spatially homogeneous Boltzmann equation

$$\begin{cases} \dfrac{\partial}{\partial t} f(t, v) = Q(f, f)(t, v) \\ f(0, v) = f_0(v) \end{cases} \tag{1}$$

where $f(t, v)$ for each t gives the distribution of velocities of a spatially homogeneous gas. The operator $Q(f, f)$ in the right hand side is the collision term, which here is assumed to describe hard sphere interaction. The details are given below. It is known that if

$$\int_{\mathbb{R}^3} f_0(v)(1 + |v|^2)\, dv < \infty$$

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then there is a unique energy-conserving solution:

$$E(f(t,\cdot)) \equiv \int_{\mathbb{R}^3} f(t,v)\,|v|^2\,dv = E(f_0) = \text{const}$$

for all $t > 0$. This solution also conserves mass and momentum:

$$\int_{\mathbb{R}^3} f(t,v)\,dv = \text{const}$$

$$\int_{\mathbb{R}^3} f(t,v)\,v_i\,dv = \text{const}, \qquad i = 1, 2, 3$$

Results of existence and uniqueness have been proven under different additional assumptions by e.g. Carleman [Ca], Arkeryd [Ar] and with no assumptions in addition to the conservation of energy in [MW]; numerous references can be found in the latter.

In all proofs of uniqueness the conservation of energy is *assumed* to hold, either implicitly by requiring the solution to satisfy moment conditions strong enough to imply the conservation of energy (see, e.g., [Gu]), or explicitly. It is also proven, first in [MW], and then, independently in [Lu], that the energy of any solution of the Boltzmann equation is non-decreasing.

However, as demonstrated in this note, it is quite possible that the energy is strictly increasing, and hence uniqueness does not hold, unless one restricts the class of solutions to the physical ones for which energy is conserved. This seems to be a general statement. There are *always* solutions for which the energy is not conserved.

The example that is constructed in Section 3 below can be formulated as follows:

> There is a solution $f(t,v)$ to (1) with $f_0(v) = (2\pi)^{-3/2}\exp(-|v|^2/2)$ such that for any $t > 0$, the energy $E(f(t,\cdot)) = 2E(f_0)$.

Clearly, the unique solution that conserves energy is f_0 itself.

All this holds for the spatially homogeneous case, and only for hard potentials (not including the so-called Maxwellian molecules). In order to simplify the notation a little in this note, only the case of hard spheres is considered. General hard potentials are treated in the same way. The details of this and a more detailed study of the behavior of the energy increase are treated in [LW].

A more challenging problem, but perhaps also a more interesting one, is to study the implication of this to the general case with a space dependent solution.

The construction that is carried out in Section 3 relies on *a priori* estimates of solutions to the Boltzmann equation for hard potentials. These estimates can essentially be found already in [W], but are given the more precise form needed for this note in Section 2.

2. SOLUTIONS OF THE BOLTZMANN EQUATION AND MOMENT ESTIMATES

The collision operator in the right hand side of (1) is different depending on the particular molecular interaction that is being considered. In the present note only the case of hard spheres is considered (for the more general case of hard potentials, see [LW]), and then

$$\begin{aligned} Q(f, g)(v) &= \int_{\mathbb{R}^3} \int_{S^2} (f(v')\, g(v_1') - f(v)\, g(v_1))\, |v - v_1|\, d\omega\, dv_1 \\ &= Q^+(f, g)(v) - f(v)\, Lg(v) \end{aligned} \tag{2}$$

This is the form of Q that is obtained when parametrizing the post-collisional velocities v' and v_1' according to

$$\begin{cases} v' = \dfrac{v + v_1}{2} + \dfrac{|v - v_1|}{2}\,\omega \\[2ex] v_1' = \dfrac{v + v_1}{2} - \dfrac{|v - v_1|}{2}\,\omega \end{cases} \tag{3}$$

The main theorem in [MW] contains the following result.

Theorem 1. Let $f_0(v)$ be a positive function with $\int_{\mathbb{R}^3} f_0(v)\, dv = 1$. There is a unique positive function $f(t, v)$ that solves (1) and conserves the energy, $E(f(t, \cdot))$. This solution satisfies the following:

1. Let $t_0 > 0$. For all $s > 0$ and all $t \geqslant t_0$, $Y_s(t) \equiv \int_{\mathbb{R}^3} f(t, v)\, |v|^s\, dv \leqslant C_{s, t_0}$.
2. The function is continuous with values in L^1, i.e., $f \in C([0, \infty[; L^1(\mathbb{R}^3))$.
3. If the entropy of the initial data is bounded, i.e., if $H(f_0) \equiv \int_{\mathbb{R}^3} f_0(v) \log f_0(v)\, dv$ is bounded, then $H(f(t, \cdot))$ is non-increasing.

Remark 1. Entropy is not considered in [MW] because the aim there is to study solutions with the weakest possible conditions on the initial data. A proof of point 3 of Theorem 1 can be found e.g. in [Ar].

Remark 2. The main problem when dealing with the homogeneous Boltzmann equation is the fact that the factor $|v - v_1|$ in $Q(f, g)$ makes the collision operator unbounded. The estimate of point 1 is due to the fact that the unboundedness helps to control higher moments Y_s. Point 1 implies that actually f is smooth with respect to t: $f \in C^\infty(]0, \infty[; L^1(\mathbb{R}^3))$. For the so-called Maxwellian molecules, the collision operator is bounded, and then the analysis carried out in this note does not apply.

The key estimates needed for the construction in Section 3 are the uniform bounds on the entropy (point 3 of Theorem 1), and the moment estimate. The so-called Gibbs lemma implies that the entropy of a function is bounded from below by a constant depending on the mass and energy of the function, and therefore point 3 in Theorem 1 implies that the entropy is bounded from above and below, uniformly in time, by constants that depend only on mass, energy and entropy of the initial data. As a matter of fact, it is not essential that the entropy is bounded, but some form of local uniform integrability is needed, and the control over the entropy provides conveniently such an estimate.

The remaining part of Section 2 aims at proving that also the estimate of point 1 depends only on mass, energy and entropy of the initial data.

Theorem 2. Let $f(t, v)$ be the unique solution to (1) that conserves energy. Then, for all $s > 2$,

$$Y_s(t) \equiv \int_{\mathbb{R}^3} f(t, v)\, |v|^s\, dv \leqslant (A_{E_0, H_0} t)^{2-s} + B_{E_0, H_0, s} \tag{4}$$

the constants A_{E_0, H_0} and $B_{E_0, H_0, s}$ depend only on the mass, the energy and the entropy of the initial data.

Proof. The proof is only a more precise formulation of the corresponding estimate in [W] (see also [Bo]). After multiplying Eq. (1) by $|v|^s$ and integrating over $\mathbb{R}^3$, it is possible to change variables in the right hand side so as to obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} f(t, v)\, |v|^s\, dv \\ &\quad = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, v)\, f(t, v_1)\, |v - v_1| \int_{S^2} (|v'|^s + |v_1'|^s - |v|^s - |v_1|^s)\, d\omega\, dv_1\, dv \end{aligned} \tag{5}$$

This is a standard calculation, which is described for example in [CIP]. Note that the estimates given in Theorem 1 already imply that this change of variables is allowed. The calculation is made to give the precise estimate stated in Theorem 2. Moment estimates for the Boltzmann equation are usually obtained by using so-called Povzner inequalities, which give bounds for the integral with respect to S^2 in (5). One such inequality may be found in [MW]: There are positive constants C_1 and C_2 such that

$$\int_{S^2} (|v'|^s + |v_1'|^s - |v|^s - |v_1|^s)\, d\omega \leqslant (s-2)(C_1(|v|\, |v_1|)^{s/2} - C_2(|v|^s + |v_1|^s)) \tag{6}$$

On more inequality is needed: There are constants C_3 and C_4 such that

$$C_3(1+|v|) \leqslant \int_{\mathbb{R}^3} f(t, v_1)\, |v - v_1|\, dv_1 \leqslant C_4(1+|v|) \tag{7}$$

The constants C_3 and C_4 depend on the energy and entropy (or some other integrability estimate) of f, and therefore they can be taken to hold uniformly in time for a solution to the Boltzmann equation.

For the remaining part of the proof, write $s = 2+\gamma$. Using (6) and (7) in the right hand side of (5) gives

$$\frac{d}{dt} Y_{2+\gamma}(t) \leqslant -\gamma \bigg(C' \int_{\mathbb{R}^3} f(t, v)\, |v|^{2+\gamma+1}\, dv$$
$$- C'' \int_{\mathbb{R}^3} f(t, v)\, |v|^{1+\gamma/2+1}\, dv \int_{\mathbb{R}^3} f(t, v_1)\, |v_1|^{1+\gamma/2}\, dv_1 - C''' \bigg)$$

where C', C'' and C''' are constants only depending on mass energy and entropy of the function f. For the application of this theorem in Section 3, any $\gamma > 0$ suffices, and to abbreviate the calculation a little, only the case $\gamma \leqslant 1$ is considered; the general case is treated similarly.

Using Jensen's inequality one may deduce

$$\frac{1}{E(f(t,\cdot))} \int_{\mathbb{R}^3} f(t, v)\, |v|^{2+\gamma+1}\, dv \geqslant \left(\frac{1}{E(f(t,\cdot))} \int_{\mathbb{R}^3} f(t, v)\, |v|^{2+\gamma}\, dv \right)^{(1+\gamma)/\gamma}$$
$$= \left(\frac{Y_{2+\gamma}}{E(f(t,\cdot))} \right)^{(1+\gamma)/\gamma}$$

and similarly

$$\frac{1}{E(f(t,\cdot))} \int_{\mathbb{R}^3} f(t, v)\, |v|^{2+\gamma+1}\, dv \leqslant \left(\frac{Y_{2+\gamma}}{E(f(t,\cdot))} \right)^{1/2}$$

In summary, the moment $Y_{2+\gamma}(t)$ satisfies a differential inequality,

$$\frac{d}{dt} Y_{2+\gamma}(t) \leqslant -\gamma C' Y_{2+\gamma}(t)^{1+1/\gamma} + \gamma C'' Y_{2+\gamma}(t)^{1/2} + \gamma C'''$$

where the constants may be numerically different from previously, but still depend only on the mass, the energy and the entropy of the function f. If

$$Y_{2+\gamma} \geqslant 4 \max((C''/C')^{2\gamma/(2+\gamma)}, C'''^{\gamma/(\gamma+1)}) \tag{8}$$

then

$$\frac{d}{dt} Y_{2+\gamma}(t) \leqslant -\frac{\gamma C'}{2} Y_{2+\gamma}(t)^{1+1/\gamma}$$

and this in turn implies that

$$Y_{2+\gamma}(t) \leqslant (C't/2 + Y_{2+\gamma}(0)^{-1/\gamma})^{-\gamma} \leqslant (C't/2)^{-\gamma} \tag{9}$$

The last member is independent of the initial data, and so the proof can be concluded by combining (8) and (9).

3. NONCONSERVATION OF ENERGY FOR A SOLUTION TO THE BOLTZMANN EQUATION

Here a function $f(t, v)$ is constructed, which solves the Boltzmann equation, and for which the energy

$$E(f(t, \cdot)) = \int_{\mathbb{R}^3} f(t, v) \, |v|^2 \, dv$$

makes a jump at time $t = 0$. This is the simplest example one can think of, but a more careful construction could give many variations of this, such as solutions with continuously increasing energy (cf. [LW])

Let

$$M(v) = (2\pi)^{-3/2} \, e^{-|v|^2/2}$$

This is the Maxwellian with mass 1 and energy, $E(M) = 3$. The Maxwellians are equilibria for the Boltzmann equation and hence the unique energy-conserving solution to (1) with $f_0 = M$ is M itself. Consider now a sequence of positive functions $g_n(v)$ that satisfy

$$\int_{\mathbb{R}^3} g_n(v)\,dv \to 0 \qquad \text{as} \quad n \to \infty$$

$$\int_{\mathbb{R}^3} g_n(v)\,|v|^2\,dv = 3 \qquad \text{for all} \quad n$$

$$\int_{\mathbb{R}^3} (M(v) + g_n(v)) \log(M(v) + g_n(v))\,dv$$

$$\to \int_{\mathbb{R}^3} M(v) \log(M(v))\,dv \qquad \text{as} \quad n \to \infty$$

One example is $g_n = (3/4\pi n^{-5}) \exp(-|v|/n)$.

Next let $f_n(t, v)$ be the unique solution of (1) that conserves energy and for which

$$f_n(0, v) = M(v) + g_n(v)$$

The uniform bound on $\int_{\mathbb{R}^3} f_n(t, v)(1 + |v|^2 + |\log f_n(t, v)|)\,dv$ implies that for almost every t, $f(t, \cdot)$ belongs to a weakly compact set of L^1, and it is possible to extract a subsequence that converges weakly for all t belonging to dense subset of $\mathbb{R}^+$. The uniform bound on the energy implies that the limit can be extended to a continuous, and even differentiable function

$$f(t, v) \in C([0, \infty[; L^1(\mathbb{R}^3))$$

Using the bounds of moments of f_n and f, it is easy to see that $Q(f_n, f_n)$ converges weakly in L^1 to $Q(f, f)$, and hence $f(t, v)$ is a solution to the Boltzmann equation. In fact, this procedure is the usual way of constructing solutions to the Boltzmann equation (see [Ar]). What was new in [MW] is that this can always be done so that the energy is conserved for the solution, and that there is only one such solution.

Remark 3. It is sufficient here to consider a weakly compact set of functions f_n, but for solutions to the homogeneous Boltzmann equation, the sequence f_n actually belongs to a strongly compact set. This is due to the fact that the positive part of the collision operator is regularizing.

Looking more carefully now at the limiting function f, one can see that at $t=0$, the weak limit is $M(v)$, and so the energy at $t=0$ is equal to 3. However, the situation is different for $t>0$. For any t positive,

$$\int_{|v|<R} f(t,v)\,|v|^2\,dv = \int_{|v|<R} (f(t,v)-f_n(t,v))\,|v|^2\,dv + \int_{\mathbb{R}^3} f_n(t,v)\,|v|^2\,dv - \int_{|v|>R} f_n(t,v)\,|v|^2\,dv$$

The middle term is equal to $E(M+g_n)=6$ independently of n, and the last term converges to zero as $R\to\infty$ independently of n because of the uniform bounds given by Theorem 2:

$$\int_{|v|<R} f(t,v)\,|v|^2\,dv \leqslant R^{2-s}\int_{|v|>R} f_n(t,v)\,|v|^2 \leqslant R^{2-s}((At)^{2-s}+B)$$

where the constants in the right hand side depend only on the mass, energy and entropy, and hence can be taken to be independent of n. Finally, the first term converges to zero for any fixed R because of the weak convergence of f_n. Now letting $R\to\infty$ and $n\to\infty$, it follows that for any $t>0$,

$$E(f(t,\cdot)) = 6 = 2E(f(0,v))$$

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REFERENCES

[Ar] L. Arkeryd, On the Boltzmann equation, *Arch. Rational Mech. Anal.* **34**:1–34 (1972).

[Bo] A. V. Bobylev, Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems, *J. Statist. Phys.* **88**(5/6):1183–1214 (1997).

[Ca] T. Carleman, *Problèmes mathématiques dans la théorie cinétique des gaz* (Almqvist & Wiksell, Uppsala, 1957).

[CIP] C. Cercignani, R. Illner, M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer Verlag, New York, 1994).

[Gu] T. Gustafsson, L^p-estimates for the nonlinear spatially homogeneous Boltzmann equation, *Arch. Rational Mech. Anal.* **92**:23–57 (1986).

[Lu] X. G. Lu, Conservation of energy, entropy identity and uniqueness for the spatially homogeneous Boltzmann equation, preprint (Department of applied mathematics, Tsinghua University, Beijing, 1998).

[LW] X. G. Lu, B. Wennberg, in preparation.

[MW] S. Mischler, B. Wennberg, On the spatially homogeneous Boltzmann equation, preprint (to appear in *Ann. de l'IHP*) (1996).

[W] B. Wennberg, Entropy dissipation and moment production for the Boltzmann equation, *J. Stat. Phys.* **86**(5/6):1053–1066 (1997).

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